

During the early planning phase of *Everyday Mathematics 4*, a team of writers worked together to read and summarize the current body of research about spatial measurement. The information that follows is the summary of that research. For more information about the planning phases of *Everyday Mathematics 4*, see the paper “*Everyday Mathematics and the Writing Process*”.

Summary of the Spatial Measurement Research in *Everyday Mathematics 4*

In order to strengthen the measurement content and pedagogy throughout *Everyday Mathematics*, we explored some of the research related to how elementary school children learn about spatial measurement—finding the lengths, areas, and volumes/capacities of geometric or real life objects. This research summary covers work produced by Battista; Clements and Sarama; Ngyuen; Outhred and Michelmore; Solomon, Vasilyeva, Levine, and Huttenlocher; and Smith, among others. Battista’s research involved testing elementary aged children to determine the order in which skills necessary to perform various measurement tasks develop. Clements and Sarama also conducted research to establish developmental sequences, or learning trajectories, for the development of measurement skills and understandings. Barrett and Battista made a comparison of the trajectories for linear measurement developed by Clements and Sarama and by Battista. Nguyen posits theoretical trajectories for development of length and area understanding. Outhred and Michelmore focus on skills and understandings children develop before receiving formal instruction in area. Solomon, Vasilyeva, Levine, and Huttenlocher assessed children’s procedural and conceptual understanding of measurement by testing kindergarteners’ and second graders’ abilities to measure length using either a ruler or a row of discrete items. Smith’s research has identified conceptual elements related to measurement that are important, but often missing, from current curricula and teaching.

The limitations of these studies are that each is fairly narrow in scope and disconnected from the others. The studies regarding length either have a conceptual focus and ignore ruler/tool usage or they focus entirely on measuring with rulers, with some attention to how children’s proficiency or difficulty with this skill illustrates conceptual understanding (or lack thereof). The studies that attend to both conceptual and procedural aspects of measurement do not posit a clear order or approach for layering conceptual understanding with measurement practice. The studies exploring area exclusively tested children on finding the areas of rectangles, and thus do not shed light on finding the areas of other figures and how this type of work might fit in with understanding and measuring area more generally. Additional informal discussions with Jack Smith and his team have helped us to bridge some of these gaps in the research, as well as address some specific questions that our working group raised.

Below, we divide spatial measurement into length, area, and volume/capacity, highlighting parallels and connections among the learning trajectories for each dimension. The majority of measurement research we reviewed has been dedicated to length, but some researchers note that measuring area and volume/capacity involve skills and concepts that parallel length-measurement skills—especially the fundamental notion of iterating units for any dimension (Battista, 2004; Battista, 2006; Outhred, et al., 2000; Smith, et al., 2008). That said, Jack Smith argues that in order to understand what area and volume/capacity are, to best understand area and volume calculations, and to be able to solve problems involving area and volume/capacity, children must explicitly explore and learn the concepts related to these topics, just as they do for length. For example, Smith advises that just because children have learned about iterating units

for length does not mean they will automatically realize that units are also iterated for area or volume/capacity. So, the concepts that run in parallel for length, area, and volume/capacity must be made explicit and repeated for each dimension; they must also be connected with one another.

Length

Research indicates that even very young children typically have the ability to recognize objects as larger or smaller without attending to a particular dimension. Before age five, they are able to recognize length as a dimension separate from other measurable attributes of objects such as area, volume/capacity, or weight (Barrett, et al., 2011; Nguyen, 2010). At this point, they begin to make holistic comparisons of lengths based on how things look, without making direct comparisons (Barrett, et al., 2011; Battista, 2006; Clements, et al., 2006; Nguyen, 2010; Smith, et al., 2008). Next, they start to refine vague visual comparisons of lengths into direct comparisons (Barrett, et al., 2011; Battista, 2006; Clements, et al., 2006; Nguyen, 2010; Smith, et al., 2008). Not only will they look at two paths and say that one looks a little longer than the other, but they may also try to line up the ends of the paths (Barrett, et al., 2011; Battista, 2006; Clements, et al., 2006; Nguyen, 2010; Smith, et al., 2008). These qualitative comparisons of length seem to be an important early step toward understanding quantitative measurement. Being able to say that one object is shorter than another allows children to then use the shorter object as a unit for measuring the longer one. (Clements, et al., 2006; Smith, et al., 2008)

Just as holistic, visual comparisons of length serve as a precursor to direct comparisons of length, counting to find length—regardless of unit iteration—serves as a precursor for quantitative measurement that involves correct unit iteration (Clements, et al., 2006). For example, children may count as they move their fingers along a path or count unevenly spaced, differently sized beads on a string in an attempt to measure the lengths of the path and string (Clements, et al., 2006). When they do this, they begin to see that greater measures mean longer lengths (an idea that is embedded in the language of number use, ten is *bigger* than three, eight is *higher* than five) (Clements, et al., 2006; Smith, et al., 2008). Battista considers understanding length as a measurable attribute of objects, qualitative comparisons of lengths, and counting in the context of length to be pre-measurement skills.

Research suggests that at around age five, children start to intuitively understand length conservation—that length is conserved through rigid transformations, as well as through breaking and reassembling or partitioning; and that length is independent of the measurement unit used (Clements, et al., 2006; Nguyen, 2010; Smith, et al., 2008). Children can use these ideas when comparing two lengths because they can then decompose and recompose paths (to match each other or straighten both, for instance) or point to specific parts of paths when they compare total lengths of paths (Clements, et al., 2006). Children at this stage understand that given any two paths, either one is longer than the other or they are equal in length (Smith, et al., 2008). They begin to explore the idea that length has transitivity as they compare two lengths by representing them with a third (Barrett, et al., 2011; Battista, 2006; Clements, et al., 2006; Smith, et al., 2008).

Representing the length of an object with another object is where unit iteration comes into play (Barrett, et al., 2011; Battista, 2006; Clements, et al., 2006; Smith, et al., 2008). At first, children begin by iterating what they consider to be a unit length along an object or path, but they make errors related to not understanding that units must be the same length or not knowing how to

enumerate units properly (without gaps, overlaps, etc.) (Barrett, et al., 2011; Battista, 2006; Clements, et al., 2006; Smith, et al., 2008). Next children work toward correct unit iteration, when they can position unit lengths end-to-end along an object and count them to arrive at the measure (Barrett, et al., 2011; Battista, 2006; Clements, et al., 2006; Smith, et al., 2008). Several researchers caution that this is an area where children's conceptual understanding is often lacking, noting that children tend to rely on procedural steps that they may not fully understand (Levine, et al., 2009; Smith, 2008; Smith, et al., 2008; Solomon, et al., n.d.). Smith and others advise that, at this stage, measurement activities should develop conceptual understandings and correct procedures through examples and discussions of incorrect and non-identical unit iteration, in addition to having children practice correct unit iteration techniques. These same researchers suggest the importance of helping children see the connection between measuring by concretely iterating units and measuring with rulers or other tools that include pre-iterated unit markings. They note that layering units on or alongside standard measuring tools (or constructing such tools from standard unit lengths) may promote children's understanding that the marks on standard measuring tools represent iterations of standard length units (Levine, et al., 2009; Smith, 2008; Smith, et al., 2008; Solomon, et al., n.d.). Studies of current curricula, including *EM*, indicate that there is not adequate attention to this approach (Smith, 2008; Smith, et al., 2008).

Relatedly, Smith's data on how measurement is commonly taught indicates that when children first begin to measure with a ruler, they are typically instructed to align one end of the length to be measured with the zero mark on the ruler, look for the number that is closest to the other end ($= x$), and state that the measured length is x units. Data from Solomon, Vasilyeva, Levine, and Huttenlocher, as well as from Smith, caution against this practice, though. Their findings suggest that, even when objects are misaligned with the zero end of the ruler, children often incorrectly use the technique of reading the number at the right edge of the object being measured, reflecting an over-reliance on a measurement procedure they don't fully understand (Smith, 2008; Smith, et al., 2008; Solomon, et al., n.d.).

These researchers suggest that it may be preferable to teach children how to measure an object with a ruler *without* aligning the zero mark with the end of the object. They further suggest that this should be done in conjunction with the types of activities described above in which children connect concrete unit iteration with the unit markings on standard measuring tools (Levine, et al., 2009; Smith, 2008; Smith, et al., 2008; Solomon, et al., n.d.).

Smith suggests that it does not seem to matter whether metric or U.S. customary units and tools are introduced first, but that it is best to introduce them separately. Smith believes that children should have an opportunity to solidify their knowledge and experience in one system of measurement before encountering a different system. He recommends beginning with U.S. customary units, largely because children may already have some experience with it. As children use U.S. customary or metric measures, they will likely become familiar with and properly use abbreviations for the measures (Smith, et al., 2008).

At about age eight, children can engage in "conceptual ruler measuring," or estimating lengths of unpartitioned objects by imagining measuring them, rather than by iterating units or using measuring tools (Barrett, et al. 2011). At this stage, children with an understanding of unit iteration and counting can imagine a ruler or their wingspans along the wall of a classroom (Barrett, et al. 2011; Clements, et al., 2006; Smith, et al., 2008). Children in primary grades are

able to define the units they use (standard and non-standard) with actual size drawings, scale drawings, or verbal statements (Smith, et al., 2009).

Understanding that a path is both a set of added lengths and a single length, and that those two lengths have the same measure, is called “integrated conceptual path measuring” (Barrett, et al. 2011). It is developed and practiced by adding iterations of units³. Battista shows this in activities where children find perimeters of rectangles by counting and adding lengths of sides (Battista, 2004). This understanding that length is additive allows children to begin adding lengths without iterating (Barrett, et al., 2011; Battista, 2004). In Battista’s research, this entailed children finding the perimeter of a rectangle by adding the lengths of sides. Children who have achieved this understanding can see lengths as continuous and also as numbers that obey operations.

Later in the primary grades, children can compare lengths by using property-based transformations to situate objects in ways that facilitate comparison. Children use slides, flips, and turns to transform shapes, comparing perimeters by rearranging for congruency (Battista, 2004).

According to Barrett and Battista, once children understand all of the previously discussed skills, they can coordinate and integrate abstract measuring with derived units—units found by calculation rather than direct measurement (Barrett, et al. 2011). This involves converting among units, using numeric operations on lengths, and justifying significant digits. Smith specifically calls out the use of ratios to convert measures within and across measurement systems among these skills (Smith, et al., 2008).

Area

Research suggests that children’s earliest understandings about area involve distinguishing between length and area and recognizing area as an attribute of objects or shapes. According to both Nguyen and Smith, this means that children can point to the surface of an object and say that it has area and also indicate that the length of the object is a line from one end of the surface to another, without confounding the two measures (Levine, et al., 2009; Smith, et al. 2008).

Once children have an idea of what area is, they begin to understand that area, like length, is conserved over transformations. They can compare areas directly by placing one shape on top of another to check for complete covering. They may, however, have a more difficult time comparing areas that do not completely overlap until they understand conservation of area over breaking. When they understand conservation of area over breaking, children realize that area does not change when shapes are cut apart and reassembled. Understanding that area is conserved when an object is cut and rearranged into the *same* configuration is understood first, followed later by understanding that area is conserved when cutting and rearranging an object into a *different* configuration. Then children can compare areas by decomposing and recomposing shapes into configurations that can be compared directly (Nguyen, 2010).

Smith clarifies that children are able to iterate area units, tile, and partition areas once they understand conservation of area by decomposition (just as being able to compare lengths allows children to begin measuring length quantitatively). He advises that children begin measuring area by iterating, then by tiling and partitioning. When asked to cover or partition a rectangular area into squares, children learn to cover the area completely with uniform size and shape units

without overlaps or gaps (Outhred, et al., 2000). Battista calls the ability to correctly align and count iterations of units “units-locating”. Without units-locating skills that come from practice with iterating units, children may have trouble partitioning a rectangle into logically placed units (Battista, 2004). Establishing units-locating skills aligns with Outhred and Mitchelmore’s spatial structuring, wherein children align units in an array with the same number of units in each row.

Breaking an area into units allows children to more specifically compare areas using commensurate units (Nguyen, 2010). This skill draws on the idea of area transitivity, which is parallel to their experiences with length transitivity (when they compare two lengths by representing them with a third) (Nguyen, 2010). Also, just as children learn that changing the unit size does not change the length of an object, children also understand that changing unit size does not change an object’s area (Nguyen, 2010). Children begin to understand that area measure is inversely related to unit size (Nguyen, 2010; Outhred, et al., 2000).

Once iterating and tiling skills help to establish spatial structures, organizing area by composites becomes easier. According to Battista, *organizing by composites* “combines an array’s basic spatial units (squares or cubes) into more complicated composite units that can be repeated or iterated to generate the whole array” (Battista, 2004). This helps to solidify the units-locating process so that children can “see a corner square as part of a row and part of a column” (Battista, 2004). Organization by composites is most sophisticated when children structure arrays in terms of maximal composites, e.g. entire rows or columns (Battista, 2004).

Understanding that the length of a line specifies the number of unit lengths that will fit alongside it allows children to then use the lengths of sides of a rectangle to determine both the number of units in each row and the number of rows. Children can then develop and understand an area formula for rectangles that involves calculating the number of units in a rectangular array from the number of units in each row and each column – determined by the lengths of the sides (Outhred, 2000).

Area obeys numeric operations just as length does. Conservation of area over breaking begins to develop the idea of area being additive. Understanding the inverse relationship between area measure and unit size will help children understand the quantitative relationship between unit size and area (Nguyen, 2010).

Volume/Capacity

Battista’s research focuses on volume in conjunction with area; more specifically, it focuses on *units-locating* and *organizing by composites*. Just as Battista outlines for area, once children start to be able to iterate volume units correctly and logically, they then start to organize the space into composites—in this case, the composites include layers. Seeing a volume as a stack of layers helps children to focus their units-locating process and identify corners as being parts of rows, columns, and layers. The trajectory has children eventually using maximal composites—that is, stacking layers. Then children can most efficiently and effectively use units-locating and organizing by composites to determine volumes of rectangular prisms (Battista, 2004).

Just as the area research we read does not go much beyond finding areas of rectangles, the volume research focused entirely on finding volumes of rectangular prisms.

References

- Barrett, J. & Battista, M. (2011). "Comparing Two Trajectories for Linear Measurement." Presentation at NCTM Research Precession Symposium, *Examining Learning Progressions, Trajectories, and Levels: Beyond Scope and Sequence*; Indianapolis, IN, 12 April 2011
- Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning*, 6(2), 185-204.
- Battista, M. T. (2006). Understanding the development of students' thinking about length. *Teaching Children Mathematics*, 13(3), 140.
- Clements, D., & Samara, J. (2006). *Learning Trajectories for Primary Grades Mathematics*. Retrieved from <http://gse.buffalo.edu/org/triad/tbb/cca/articles/Learning%20Trajectories.pdf>
- Levine, S. C., Kwon, M. K., Huttenlocher, J., Ratliff, K., & Deitz, K. (2009). Children's understanding of ruler measurement and units of measure: A training study. In *Proceedings of the 31st Annual Conference in the Cognitive Science Society*. Cognitive Science Society.
- Nguyen, K. (2010). *Investigating the Role of Equipartitioning and Creating Internal Units in the Construction of a Learning Trajectory for Length and Area*. (Doctoral Dissertation). Retrieved from ProQuest Dissertations & Theses Global database (ProQuest document ID 853135223).
- Outhred, L. N., & Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education*, 144-167.
- Smith, J. (2008). *Memo to Andy Isaacs & the Everyday Mathematics Author Team*.
- Smith, J., Dietiker, L., Lee, K., Males, L. M., Figueras, H., Mosier, A., & Sisman, G. (2008, March). Framing the analysis of written measurement curricula. In *Annual Meeting of the American Educational Research Association*.
- Solomon, T., Vasilyeva, M., Levine S. C., Huttenlocher, J., & Ratliff, K. R. (under review). Sizing it up: What do elementary school children understand about linear measurement?