During the early planning phase of *Everyday Mathematics 4*, a team of writers worked together to read and summarize the current body of research about operations. The information that follows is the summary of that research. For more information about the planning phases of *Everyday Mathematics 4*, see the paper "*Everyday Mathematics* and the Writing Process."

Summary of Operations Research in *Everyday Mathematics* 4

Addition and Subtraction

A review of research suggests that in order to develop fluency with multidigit addition and subtraction of whole numbers and decimals, children must concurrently develop a flexible understanding of number. Two fundamentally different ways of interpreting multidigit numbers are the *sequential* understanding of number and the *composite* understanding of number (Verschaffel, Greer, and De Courte 2007). Each of these understandings of number leads to classes of addition and subtraction strategies. As children develop a flexible understanding of number and a solid grasp of place value, they can also develop a flexible repertoire of addition and subtraction strategies, which in turn leads to fluency with computation.

When children are first exposed to multidigit numbers, they have a unitary conception of these numbers. They see a group of 15 objects as one thing, and cannot meaningfully separate it into groups (Verschaffel, Greer, and De Courte 2007). Children at this stage have not yet constructed the idea of *hierarchical inclusion,* or the fact that numbers build on each other, and larger numbers are made up of smaller numbers (Fosnot and Dolk 2001b).

As children continue to work with multidigit numbers, they begin to assign meaning to parts of the number. When they hear the number 53, they understand that "fifty" refers to 50 objects and "three" refers to 3 objects. They may try to link this understanding to written numerals, but do not immediately grasp the place-value structure of numbers. Children at this stage will commonly make the error of writing "503" for 53 (Verschaffel, Greer, and De Courte 2007; NRC 2009).

As children grapple with the idea of place value in kindergarten and first grade, they are also typically engaging in rote counting activities. They count by tens and by ones, and eventually learn to coordinate counting by tens and continuing on counting by ones. They begin to see multidigit numbers as structured into groups of ten and ones (Verschaffel, Greer, and De Courte 2007; NRC 2009).

After they have learned to coordinate counting by tens and ones, they are able to start counting by tens in the middle of a decade. That is, they are first able to count 10, 20, 30, 31, 32, 33, then later able to count 14, 24, 34, 35, 36, 37. The ability to count in this manner shows the development of a *sequential* understanding of number, or an understanding that numbers build on themselves as you count up. Using this sequential understanding, children will often invent "jump" strategies for addition, where they start at one addend and jump up the number line by chunks of the second addend (Verschaffel, Greer, and De Courte 2007). Open number lines are useful tools for illustrating these strategies. Children start out by making the jumps by ten, which corresponds with the counting activities. For example, 44 + 28 might be solved by thinking 44 + 10 + 10 + 8, or 44, 54, 64, plus 8 is 72. Later, children become more efficient with the jumps; for example, they may only make one jump of 20 and one jump of 8 to add 28 (Fosnot and Dolk 2001b).

While children are developing their sequential understanding of number and constructing jump strategies for addition, they are still working to fully understand the place-value structure of numbers. While they may be able to count by tens and assign the correct value to the tens digit of a number, they may not fully understand that the groups of ten they are counting by are made up of ten ones, or be able to switch to thinking of these tens as ten ones when this strategy is useful. Herein lies an important connection between the addition/subtraction facts trajectory and this multidigit addition/subtraction trajectory. One important strategy for solving facts is the making-tens strategy, wherein children decompose one addend so that they can transform a fact into a ten plus some other number (Clements and Sarama 2009). This work helps children to understand that ten ones make up one ten.

To make this understanding useful for multidigit work, the idea of thinking of 1 ten as 10 ones must be connected to children's work with multidigit numbers. When counting by tens or hundreds or using manipulatives to represent such numbers, children may not be thinking of the quantity inherent in the tens. They may simply be using ten as a unit (Fosnot and Dolk 2001b). Activities such as having children count out a multiple of ten with manipulatives, then asking both "How many tens?" and "How many ones?" may help children to retain the sense of quantity in a set of tens (Clements and Sarama 2009). While being able to construct a unit of ten out of 10 ones is critically important, children need to understand that when they trade 10 ones for 1 ten, they are constructing a new unit that is convenient to count by, but not somehow making those 10 ones disappear. They must be able to switch back to thinking about a ten as 10 ones when that strategy is useful. Children can typically achieve this understanding by the end of Grade 1.

When this goal is achieved, children have a more fully developed understanding of place value. They can think of the 5 in 53 as both 5 tens and 50 ones. Children can now think of multidigit numbers in a new way. They understand multidigit numbers not only as a sequence that builds on itself (as discussed above), but they also understand individual numbers as *composites* of other numbers. This understanding is related to the idea of hierarchical inclusion discussed above, but it has now been expanded to larger numbers. They integrate their understanding of place value with the composite understanding of number, and can split two-digit numbers into tens and ones. This leads to "split" or "decomposition" strategies for addition, where tens and ones are added separately, and then the two sums are combined for a final answer (Verschaffel, Greer, and De Courte 2007; Fosnot and Dolk 2001b; Clements and Sarama 2009). The ability to switch from thinking of ten as 1 ten to 10 ones and back becomes crucial when theses strategies are used and regrouping is required. Children can progress this far on the trajectory during the course of Grade 2.

After each of these understandings of number is developed, children progress to being able to switch flexibly among these ways of thinking about number and among these addition strategies. They use the connections between addition and subtraction to develop corresponding strategies for subtraction (Fosnot and Dolk 2001b). As children become comfortable with their repertoire of strategies, they expand the techniques to deal with larger numbers in Grade 3 (Clements and Sarama 2009).

One other research finding is relevant here: In addition to inventing jump and split strategies for addition and subtraction, children will also sometimes use compensation strategies, such as transforming the difficult 123 – 47 into the easier 126 – 50 (Verschaffel, Greer, and De Courte 2007; Clements and Sarama 2009). The ideal end of this trajectory would be for students to not only develop all three types of strategies, but also flexibly apply them in ways that make sense for the problem at hand. Split strategies are most closely related to the standard algorithms for addition and subtraction, but neither the specific algorithms nor the splitting idea in general are always the most appropriate or convenient approach to solving a problem.

Multiplication and Division

Children's understandings of multiplication and division are initially based on additive thinking. In kindergarten and first grade, students will solve multiplication problems by creating equal groups and counting all, and solve division problems by either dealing out objects or creating groups of the appropriate size and counting the groups (Verschaffel, Greer, and De Courte 2007; Clements, B7; Fosnot and Dolk 2001a). Children then move to solving multiplication problems by skip counting or repeated addition (Clements, B7; Vershaffel, Greer, and De Courte 2007; Fosnot and Dolk 2001a). They may also solve division problems by skip counting up to the dividend or repeated subtraction. This shows evidence of children's ability to think of a group as a unit (much like they think of a ten as a unit as described in the addition and subtraction trajectory) and is more sophisticated than counting all. However, it still relies of additive thinking as opposed to multiplicative thinking.

As conceptual understanding of multiplication is developed through work with equal-groups and area models of multiplication, children will show evidence of multiplicative thinking through use of doubling strategies, compensation strategies, partitioning of one factor and using the distributive property, or any other techniques where they think of more than one group at a time (Verschaffel, Greer, and De Courte 2007; Clements, B7; Fosnot and Dolk 2001a). Once this more sophisticated repertoire of multiplication strategies is developed, students can work to develop fluency with multiplication facts.

The understanding of the array model and the partitioning strategies that children develop while learning multiplication and division facts can then be extended to computation with multidigit numbers. Beginning in third grade, when children multiply 2-digit numbers by

1-digit numbers, they can partition the 2-digit number to create two different, easier multiplication problems. When the 2-digit number is less than 20, they can partition it to create two known facts. When the 2-digit number is larger, they can partition the number into tens and ones, and use times-ten shortcuts and techniques to solve the problem (Clements, B7; Fosnot and Dolk 2001a).

Partitioning techniques are extended further when children begin to solve 2-digit by 2-digit multiplication problems. Children can partition both factors, often into tens and ones, and then multiply each part of one factor by each part of the other factor. Drawing an area model to illustrate the problem can help children to keep track of the parts of the product and make sure that they have included all the parts (Fosnot and Dolk 2001a; Clements, B7). This technique is the foundation of several different whole-number multiplication algorithms. However, the culmination of this trajectory is not just fluency with these algorithms. Rather, the goal is to enable children to approach problems flexibly, decomposing them in ways that are most useful, and not necessarily going straight to an algorithm when another strategy is appropriate.

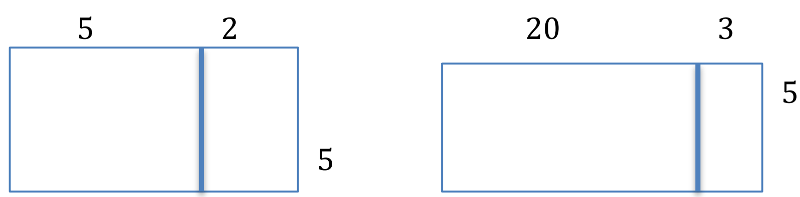
As part of the facts trajectory, students also develop an understanding of the relationship between multiplication and division. As students develop the multiplication strategies described above, they also will also learn corresponding division strategies. As they come to understand division as the inverse of multiplication, they may use building-up strategies, wherein they find the quotient by keeping track of how many divisors they need to make up the dividend (Verschaffel, Greer, and De Courte 2007). They use multiplicative understanding to move from repeated subtraction to removing "groups of groups" from the dividend. They partition the dividend into parts to make smaller problems, and sometimes will progress to using the base-ten structure of numbers as the basis for partitioning (Fosnot and Dolk 2001a; Verschaffel, Greer, and De Courte 2007). These strategies are developed throughout fourth grade. Flexibly applying these skills to division problems and extending the skills to larger numbers is the top of this trajectory.

Computational Algorithms

Research supports the value of "problematizing" mathematics and allowing children to invent their own procedures. Students who engage in activities that promote understanding of the structure of the number system and use this understanding to create their own procedures for carrying out operations are more likely to be able to adapt their procedures to solve different problems later on. They still perform as well as their peers on routine tasks, and they also make stronger connections between conceptual and procedural knowledge (Verschaffel, Greer, and De Courte 2007). Additionally, allowing children to construct their own procedures also improves their dispositions about mathematics. "Students who have invented their own correct procedures also approach mathematics with confidence rather than fear and hesitation" (NRC 2001).

Much of the research of children's strategies for multidigit computation suggests taxonomies of strategies often invented by children. Verschaffel, Greer, and De Courte (2007) and Clements and Sarama (2009) both describe jumping, splitting, and compensation as the three main strategy types for addition and subtraction. (See the addition and subtraction section above for a description of each of these categories.) Similar taxonomies exist for multiplication (skip counting/repeated addition, partitioning, and compensation) and division (repeated subtraction or addition, distribution, decomposing the dividend) (Verschaffel, Greer, and De Courte 2007).

Fosnot and Dolk (2001a) discuss the importance of moving from models *of* thinking to models *for* thinking. Said another way, they say that students should be able to generalize the methods they invent to solve other problems. Open number lines and area models can be used to illustrate and generalize the methods that students invent. Fosnot and Dolk (2001a) suggest that teachers use an area model to illustrate a student's solution to a specific multiplication problem, then help students to see that an area model can be a tool to think with to solve any multiplication problem. For example, the area model at the left below can be used to illustrate a student's method of decomposing 7 to solve 7 × 5: 7 × 5 = (5 + 2) × 5 = (5 × 5) + (2 × 5). Then, a student might think about how to decompose 23 to solve 23 × 5, and use the area model at the right below to think about the problem. Open number lines can be used analogously for addition and subtraction methods.



There is value in discussing the various algorithms that students invent, which will vary in efficiency and appropriateness for the situation. The National Research Council (2001) recommends discussing advantages of computational methods with students. They suggest focusing whole-class discussion around slightly more advanced procedures that some students are using. Those students can be given opportunities to share their procedures, and through the use of discussion and representation, together the class can make sense of why they work. To reinforce this work, it may also be helpful to discuss the advantages and disadvantages of different procedures (NRC 2001).

Campbell, Rowan, and Suarez (1998) discuss the following criteria for evaluating algorithms: Is the algorithm efficient enough to be used regularly without frustration or considerable loss of time? Is it mathematically valid? It is generalizable? Questions such as these could serve as guidelines for teachers when helping students to evaluate various methods for computation, including formal computational algorithms.

Careful consideration needs to be given to the timing and manner in which algorithms are introduced. In particular, algorithms should not be introduced too early. According to the National Research Council (2001), "When students fail to grasp the concepts that underlie procedures or cannot connect the concepts to the procedures, they frequently generate flawed procedures that result in systematic patterns of errors" (p. 196). They further state that curricula that emphasize understanding algorithms before applying them have been shown to lead to increases in understanding (NRC 2001). This demonstrates the importance of delaying the introduction of algorithms until students are adequately prepared to understand them.

Kamii and Dominick (1998) found that second and third graders who had never been taught algorithms produced higher percentages of correct answers on a three-addend addition problem than their fourth-grade peers who had been taught algorithms. Furthermore, when the no-algorithm group did produce incorrect answers, they were much more reasonable than the incorrect answers produced by the algorithm group. While it is not clear from the study whether the algorithms taught were the U.S. traditional algorithms, it seems clear that the algorithms were taught procedurally and without understanding. As the conceptual underpinnings of the U.S. traditional algorithms are difficult to unpack, this study gives reason for avoiding introducing the U.S. traditional algorithms too early.

However, Fosnot and Dolk (2001b) argue that traditional algorithms need not be avoided if, prior to their introduction, students have a repertoire of other strategies as well as a deep understanding of number relationships and operations. Older students may benefit from discussing these algorithms and why they work. Discussions of efficiency are relevant here, and understanding of the benefits of efficiency could be developed earlier during discussions of student-invented strategies (Fosnot and Dolk 2001b; Campbell, Rowan, and Suarez 1998; NRC 2001).

Introducing too many different algorithms concurrently can also be problematic. Ron (1998) presents examples of errors made when children try to combine characteristics and steps of two different algorithms. (In Ron’s examples, the two algorithms were U.S. traditional subtraction and European subtraction.) This illustrates some potential problems of introducing multiple algorithms at the same time.

It should also be noted that many computational algorithms are most easily executed when problems are presented vertically, with the digits lined up in place-value columns. Even when children successfully apply place-value understanding to correctly solve addition and subtraction problems given in word problems or horizontal form, children will often adopt a concatenated conception of number, or view multidigit numbers as strings of single digits with no regard to place value, if the same problems are presented in vertical form. This leads to computation errors (Verschaffel, Greer, and De Courte 2007). Thus, there may be reason to avoid presenting computation problems vertically, particularly when students are first learning computation methods.

Lastly, one of the dangers of teaching algorithms is that students will begin to use them blindly, without consideration of whether they are the appropriate method or whether the answers they obtain are reasonable. Whitin and Whitin (1998) suggest that asking children to write about what they notice about a problem and their predictions/estimates of the answer before attempting the problem may help children to apply the algorithms flexibly. They also suggest that writing about the algorithms after their use may help children remember and develop connections to number sense and place value.

References

Campbell, P. F., T. E. Rowan, and A. R. Suarez. 1998. "What Criteria for Student-Invented Algorithms?" In Morrow, L. J. and M. J. Kenney, eds. *The Teaching and Learning of Algorithms in School Mathematics: 1998 Yearbook.* Reston, VA: National Council of Teachers of Mathematics.

Clements, D. *Building Blocks* Learning Trajectories for Composing; Adding and Subtracting; and Multiplying and Dividing. NSF-funded project to develop curriculum materials; D. Clements, principal investigator. See http://gse.buffalo.edu/org/buildingblocks/index_2.htm

Clements, D. H., and J. Sarama. 2009. *Learning and Teaching Early Math: The Learning Trajectories Approach.* (Chapter 6: Arithmetic: Composition of Number, Place Value, and Multidigit Addition and Subtraction.) New York: Routledge.

Fosnot, C. T., and M. Dolk. 2001a. *Young Mathematicians at Work: Constructing Multiplication and Division*. Portsmouth, NH: Heinemann.

Fosnot, C. T., and M. Dolk. 2001b. *Young Mathematicians at Work: Constructing Number Sense, Addition, and Subtraction*. Portsmouth, NH: Heinemann.

Kamii, C., and A. Dominick. 1998. "The Harmful Effects of Algorithms in Grades 1-4." In Morrow, L. J., and M. J. Kenney, eds. *The Teaching and Learning of Algorithms in School Mathematics: 1998 Yearbook.* Reston, VA: National Council of Teachers of Mathematics.

National Research Council. 2001. *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, and B. Findell, eds. Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

National Research Council. 2009. *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity.* (Chapter 5: The Teaching-Learning Paths for Number, Relations, and Operations.) C. T. Cross, T. A. Woods, and H. Schweingruber, eds. Washington, DC: National Academies Press.

Ron, P. 1998. "My Family Taught Me This Way." In Morrow, L. J., and M. J. Kenney, eds. *The Teaching and Learning of Algorithms in School Mathematics: 1998 Yearbook.* Reston, VA: National Council of Teachers of Mathematics.

Verschaffel, L., B. Greer, and E. De Courte. 2007. "Whole Number Concepts and Operations." From Lester, F. K., Jr., ed. *Second Handbook of Research on Mathematics Teaching and Learning.* Reston, VA: National Council of Teachers of Mathematics.

Whitin, D. J., and P. E. Whitin. 1998.“The ‘Write’ Way to Mathematical Understanding.” In Morrow, L. J., and M. J. Kenney, eds. *The Teaching and Learning of Algorithms in School Mathematics: 1998 Yearbook.* Reston, VA: National Council of Teachers of Mathematics.